# Dynamics of a Singularly Perturbed Quadratic Family: $f_{\beta,c}(x) \equiv x^2 + c + \frac{\beta}{x}$

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## Introduction

- Dynamical systems: the study of how systems evolve in time.
  - Discrete ($x_{n+1} = f(x_n)$) vs. Continuous ($x' = f(x)$)
  - Long term behaviors
- Different examples of discrete systems:
  - Linear: $L(x) = x/2$
  - Quadratic: $Q_c(x) = x^2 + c$
  - 1-D Singular Perturbation: $f_{\beta,c}(x) \equiv x^2 + c + \frac{\beta}{x}$
  - 2-D Singular Perturbation (z=x+iy): $f_{\beta,c}(z) \equiv z^2 + c + \frac{\beta}{z}$

## Terminology & Properties

- An initial condition $x_0$ generates a sequence called an *orbit*.

$$x_0 - x_1 - x_2 - x_3 - x_4 - \ldots$$

where $x_{n+1} = f(x_n)$

- The goal of dynamical system is to determine the fate of all orbits.
- Special bounded orbits:
  - Fixed points: $f(x) = x$
  - Period-n-points: $f^n(x) \equiv f(f(\ldots f(x))) = x$
  - Prefixed points:

$$x_0 - x_1 - x_2 - \ldots - x_n - x_n - \ldots$$

- Critical orbit: the orbit starting from the critical point:
  - The value $p$ is a critical point if $f'(p) = 0$
  - Crtical value is $f(p)$
- The reference line: $f(x) = x$

## A Simple Linear Example

- All orbits approach 0.

$$8 - 4 - 2 - 1 - 1/2 - \ldots - 0$$

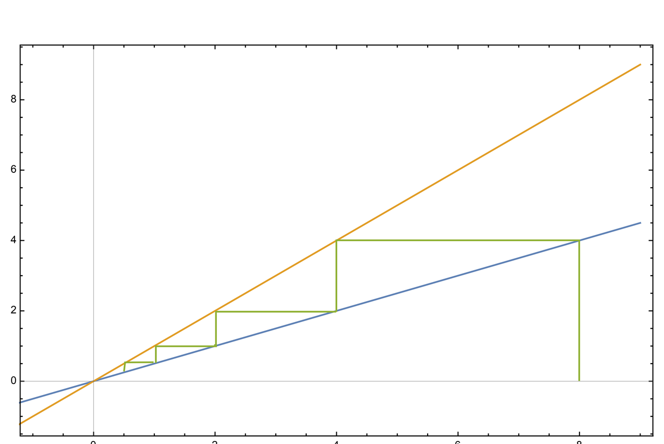
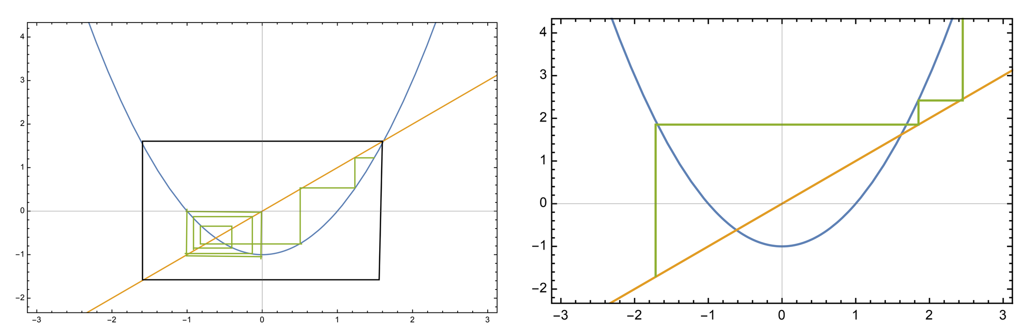


Figure 1: A graphical iteration for $L(x) = x/2$ starting from $x = 8$. Plot of $x_{n+1}$ vs $x_n$

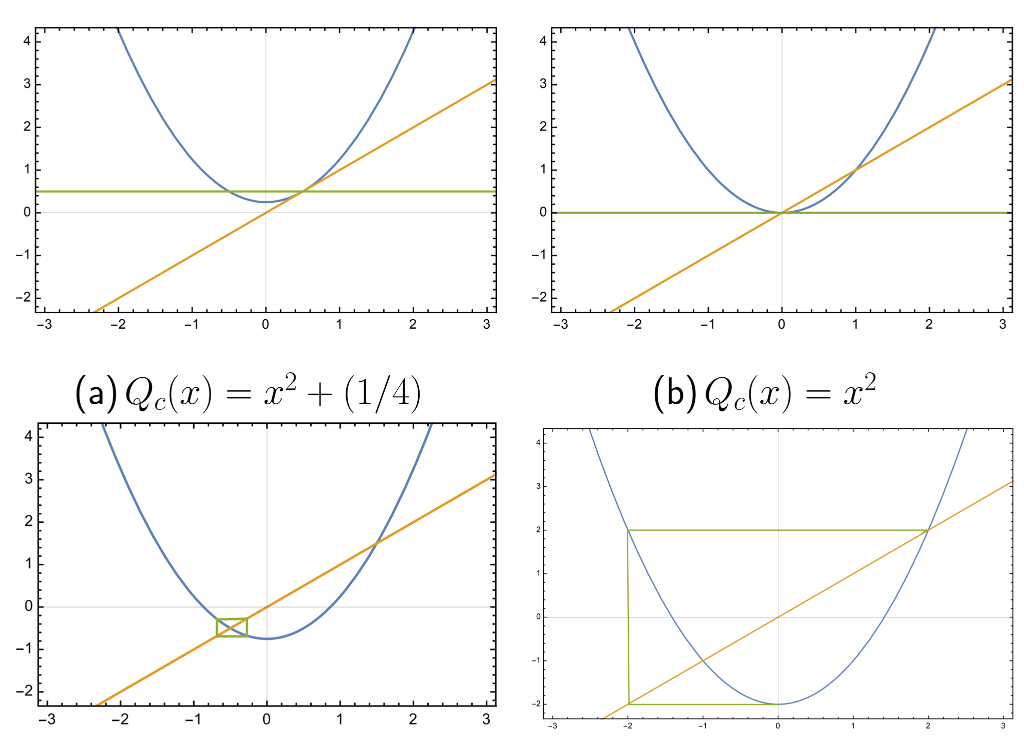
## A Quadratic Family



(a) Bounded Orbit (b) Escape Orbit

Figure 2: Graphical iteration for $Q_{(-1)}(x) = x^2 - 1$

### Key Parameters



(a) $Q_c(x) = x^2 + (1/4)$ (b) $Q_c(x) = x^2$

(c) $Q_c(x) = x^2 + (-3/4)$ (d) $Q_c(x) = x^2 + (-2)$

Figure 3: Bifurcations in the $Q_c(x) = x^2 + c$

### Orbit Diagram

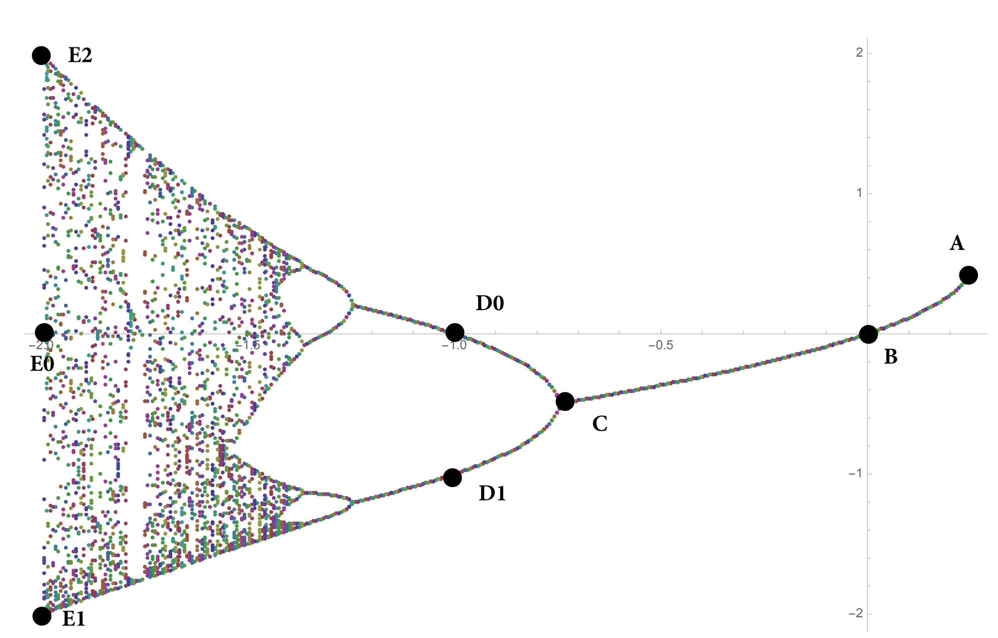
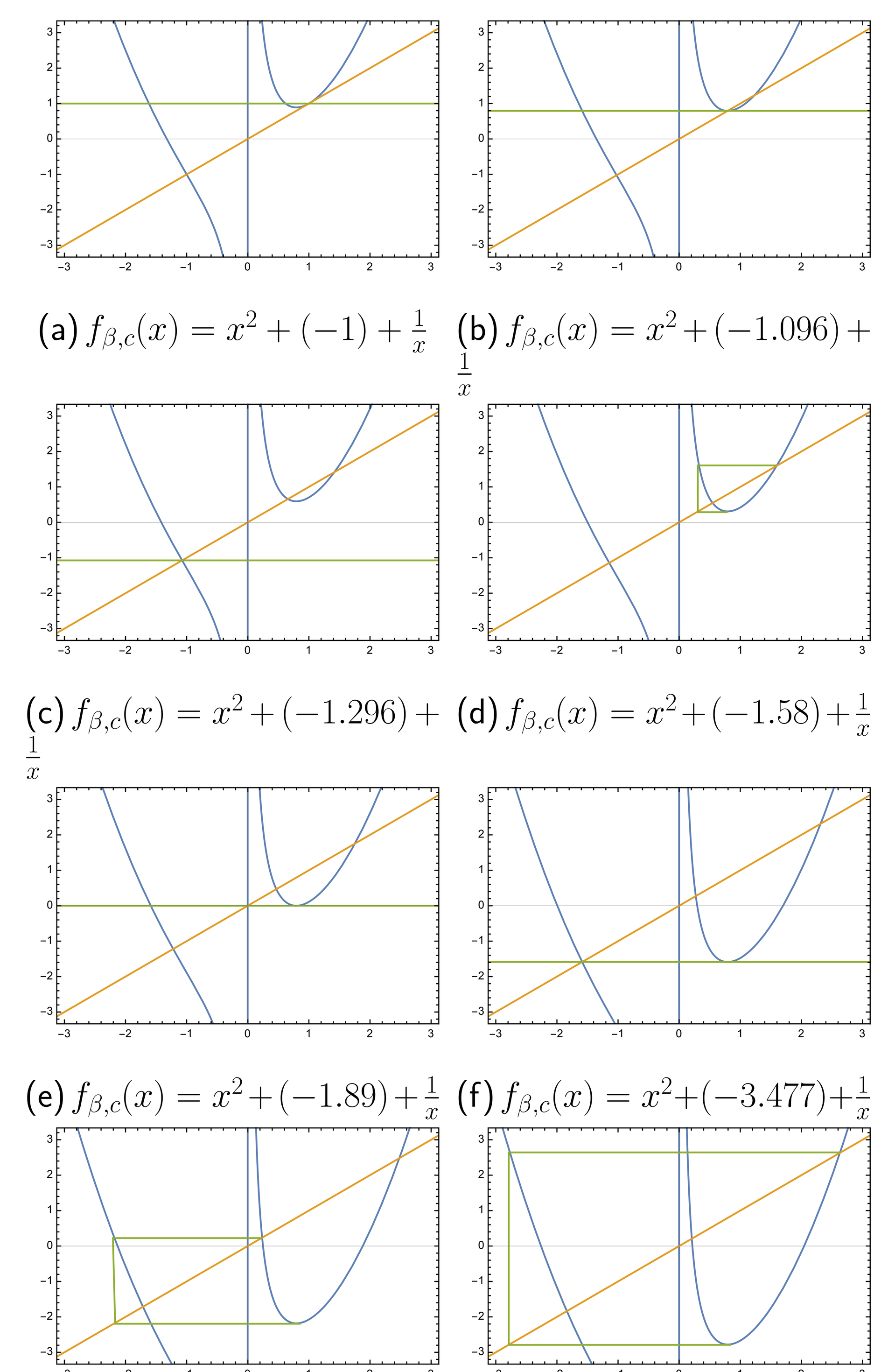



Figure 4: An orbit diagram for $Q_c(x) = x^2 + c$ Plot of $x$ vs $c$

- Fix parameter $c$.
- Start from critical point $x = 0$.
- Skip the first 160 iterations of each points
- Plot next 40 iterations

## 1-D Singular Perturbation

### Key Parameters



(a) $f_{\beta,c}(x) = x^2 + (-1) + \frac{1}{x}$ (b) $f_{\beta,c}(x) = x^2 + (-1.096) + \frac{1}{x}$

(c) $f_{\beta,c}(x) = x^2 + (-1.296) + \frac{1}{x}$ (d) $f_{\beta,c}(x) = x^2 + (-1.58) + \frac{1}{x}$

(e) $f_{\beta,c}(x) = x^2 + (-1.89) + \frac{1}{x}$ (f) $f_{\beta,c}(x) = x^2 + (-3.477) + \frac{1}{x}$

(g) $f_{\beta,c}(x) = x^2 + (-4.08) + \frac{1}{x}$ (h) $f_{\beta,c}(x) = x^2 + (-4.67) + \frac{1}{x}$

Figure 5: Bifurcation for $\beta = 1$

### Orbit Diagram for 1-D Singular Perturbation for c with $\beta = 1$

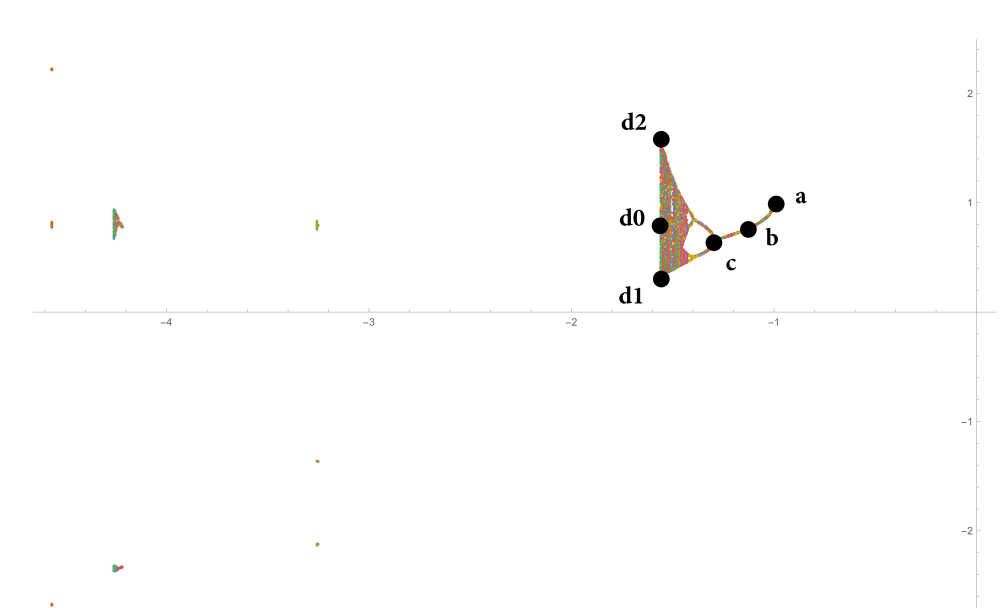



Figure 6: An Orbit Diagram for $c$ in $f_{\beta,c}(x) = x^2 + (c) + \frac{1}{x}$. Plot of $x$ vs $c$. Compare with Figure 4

## Parameter Plane for $f_{\beta,c}(x) = x^2 + (c) + \frac{\beta}{x}$

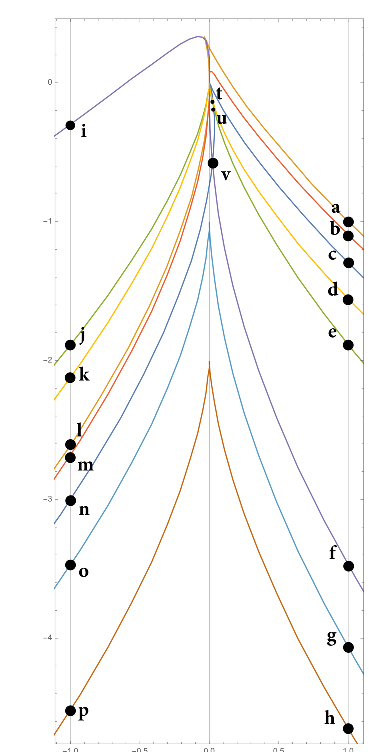
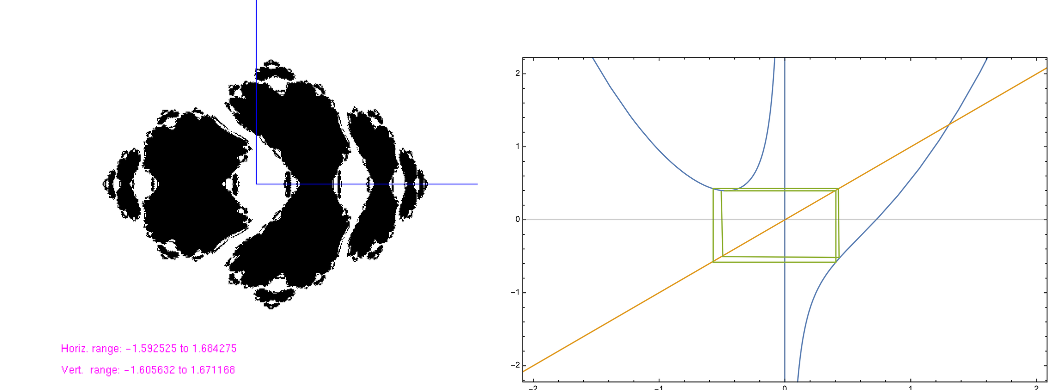


Figure 7: Parameter plane for $\beta$ and $c$. Plot of $c$ vs $\beta$. Compare with Figures 5 and 6

## 2-D Generalization



(a) Complex case: ($z = x + iy$) where black: bounded and white: escape. Plot of $y$ vs $x$. (b) Real case: (z=x). Plot of $x_{n+1}$ vs $x_n$

Figure 8: $f_{\beta,c}(z) = z^2 + (-0.25) + \frac{-0.2}{z}$

## References

[1] Robert L. Devaney.
*A First Course in Chaotic Dynamical Systems: Theory and Experiment.*
MA: Addison-Wesley, 1992.

## Acknowledgements

Further details will be provided at Math 5260 Dynamical Systems by Prof. Bruce Peckham

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